Quadratic duality and Koszul duality

References: [Mazorchuk, Ovsienko, Stroppel: Quadratic duals, Koszul dual functors, and applications]

· [Beilinson, Ginzburg, Soergel: Koszul duality patterns in Representation theory]

1) Setup / Recollection

$k = \bar{k}$; $A = \bigoplus_{i\in\mathbb{N}} A_i$ positively graded k-algebra.

Assume A_0 semisimple, in fact we'll assume $A_0 \cong k\times\dots\times k$, and A_i is f.d. k-VS.

A-mod: category of f.d. A-modules

A-gmod: category of graded A-modules with f.d. graded pieces; morphisms ~~of deg~~ preserve degree.

Def: Let $m: A_1\otimes A_1 \longrightarrow A_2$ be the multiplication in A.

Consider $m^*: A_2^* \longrightarrow (A_1\otimes A_1)^* \cong A_1^*\otimes A_1^*$ (where $V^* := \mathrm{Hom}_k(V,k)$)

Then $A^! := T(A_1^*)/(\mathrm{im}\, m^*)$ is the quadratic "dual" of A

Rem: If A is quadratic, i.e. $A \cong T_{A_0}(A_1)/(R)$, $R \subseteq A_1\otimes_{A_0} A_1$,

then $A^! = T(A_1^*)/(R^\perp)$, and $(A^!)^! \cong A$, as before.

~~Recall~~ **Goal**: Relate the representation theory of A and $A^!$ (in A-gmod)

Recall: A is called Koszul if A_0 has a linear projective resolution $\dots \longrightarrow P^1 \longrightarrow P^0 \twoheadrightarrow A_0$

"linear" means: P^{-i} is generated in degree i.

In this case: A is quadratic, $A^!$ is also Koszul, and $(A^!)^{op} \cong \mathrm{Ext}^*_{A\text{-mod}}(A^0, A^0)$ "Koszul dual".

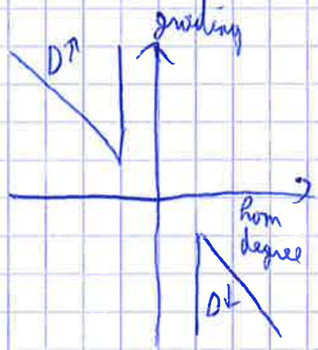
Moreover we have Koszul duality:

$$D^\uparrow(A\text{-gmod}) \cong D^\downarrow(A^!\text{-gmod})$$

("$D^\uparrow$" means that ~~an object~~ the objects of $D^\uparrow(A\text{-gmod})$ are the complexes

$\dots \longrightarrow M^i \longrightarrow M^{i+1} \longrightarrow \dots$, $M^i = \bigoplus_{j\in\mathbb{Z}} M^i_j$, satisfying $M^i_j = 0$ if $i \gg 0$ or $i+j \ll 0$

for $D^\downarrow$: $M^i_j = 0$ if $i \ll 0$ or $i+j \gg 0$)



Question:

$$\begin{array}{ccc} D^\uparrow(A\text{-gmod}) & \cong & D^\downarrow(A^!\text{-gmod}) \\ \cup & & \cup \\ ? & \overset{\cong}{\Longleftarrow} & A^!\text{-gmod} \end{array}$$

~~Recall our main "Baby" example from last time: $A=$~~

By assumption $A_0 = \underbrace{k\times\dots\times k}_{n \text{ factors}}$, get $\{e_1,\dots,e_n\}$ a $\underbrace{\text{complete}}_{\sum e_i = 1}$ set of $\underbrace{\text{orthogonal}}_{e_ie_j = 0 \text{ if } i\neq j}$ $\underbrace{\text{primitive}}_{e_i \text{ is not a nontrivial sum of idempotents}}$ $\underbrace{\text{idempotents}}_{e_i^2 = e_i}$ in A.

<u>Aside</u>: <u>Alternative viewpoint</u> [MOS]: Instead of A, we consider the graded k-linear category $\mathcal{C}$ with objects $\{1,\dots,n\}$ and morphisms $\mathcal{C}(i,j) = e_j A e_i$.

A graded $\mathcal{C}$-module is a k-linear degree-preserving functor $\mathcal{C} \to \mathrm{vect}_k$.
(and a morphism of $\mathcal{C}$-modules is a natural transformation of functors)

Then $A\text{-gmod} \simeq \mathcal{C}\text{-gmod}$ equivalence of categories. (Note $A \cong \bigoplus_{i,j} \mathcal{C}(i,j)$)

<u>Rem</u> (see [MOS]): In this setup we can easily weaken our finiteness conditions to allow ∞ many objects. This corresponds to a not necessarily unital algebra.

<u>Running examples</u>: 1) $A = \mathbb{C}[x]$; $A^! = \mathbb{C}[x]/(x^2)$; both have only 1 prim idempotent $e_1 = 1$.

2) A_Q is the path algebra of the quiver $Q = 1 \underset{b}{\overset{a}{\rightleftarrows}} 2$, modulo relation $ba = 0$; $\deg(a) = \deg(b) = 1$.
(i.e. basis $\{e_1, e_2, a = e_2 a e_1;\ b = e_1 b e_2,\ ab = e_2 ab e_2\}$)

<u>Or</u>: $\mathcal{C}_Q$ category with objects $1,2$; $\mathrm{Hom}(1,2) = \langle a\rangle_k$; $\mathrm{Hom}(2,2) = \langle e_2, ab\rangle_k, \dots$

Then $A_Q^!$ is the path algebra of Q, but with relation $ab = 0$. Hence $A_Q \cong A_Q^!$.

In fact: A_Q is Koszul (see later).

grading shift [1] homological shift

- Indecomposable projective A-g modules are given by $P(i)\langle m\rangle := Ae_i\langle m\rangle$ (for $i \in \{1,\dots,n\}$)

  $P(i) = \mathcal{C}(i,-) : \mathcal{C} \to \mathrm{vect}_k$

  (In example 1: $P(1) = \mathbb{C}[x]$, $P(1)^! = \mathbb{C}[x]/(x^2)$)

- Simple (graded) A-modules are $L(i)\langle m\rangle := \left(P(i)/P(i)_{>0}\right)\langle m\rangle$

  $$\begin{aligned} L(i)\langle m\rangle : \mathcal{C} &\to \mathrm{vect}_k \\ i &\mapsto k\langle m\rangle \\ i \neq j &\mapsto 0 \end{aligned}$$

  (In example 1: $L(1) \cong \mathbb{C}$, $L^!(1) \cong \mathbb{C}$)

- Indec injective (gr.) A-modules are $N(P(i)\langle m\rangle) \cong A^* e_i \langle m\rangle$, where $N : A\text{-gmod} \to A\text{-gmod}$, $M \mapsto A^* \otimes_A M$ Nakayama functor.

  (In example 1: $I(1) \cong \mathbb{C}[x]^*$
  $I^!(1) \cong (\mathbb{C}[x]/(x^2))^* \cong \mathbb{C}[x]/(x^2)\langle -1\rangle$)

<u>Example 2</u>: $P(1) = Ae_1 = \langle e_1, a\rangle$; $P(2) = Ae_2 = \langle e_2, b, ab\rangle$

$L(1) = \langle e_1\rangle$; $L(2) = \langle e_2\rangle$

$I(1) = \langle b^*, e_1^*\rangle$; $I(2) = \langle (ab)^*, a^*, e_2^*\rangle \cong P(2)\langle -2\rangle$

<u>Proof that A_Q is Koszul</u>: Enough to show that $L(1), L(2)$ have linear projective resolution.

$$P(1)\langle 2\rangle \xrightarrow{\cdot b} P(2)\langle 1\rangle \xrightarrow{\cdot a} P(1) \twoheadrightarrow L(1) \qquad ; \qquad P(1)\langle 1\rangle \xrightarrow{\cdot b} P(2) \twoheadrightarrow L(2)$$

$e_1 \mapsto b$, $a \mapsto ab$; $e_2 \mapsto a$; $e_1 \mapsto b$, $a \mapsto ab$; $e_2 \mapsto e_2$

$\square$

Now let's go back to our main question

2) The category $LCP(A)$ ("linear complexes of projectives")

$\nearrow$ no summands $0 \to M \xrightarrow{\sim} M \to 0$

Def: $LCP(A)$ is the full subcategory of $C(A\text{-grmod})$, whose objects are minimal complexes

$$\cdots \to X^m \to X^{m+1} \to \cdots \quad \text{s.t. } X^{-m} \cong \bigoplus_\lambda P(i_\lambda)\langle m \rangle.$$

Prop: $LCP(A)$ is an abelian category.

Proof: Suffices to check: if $X \xrightarrow{f} Y$ is a morphism in $LCP(A)$, the $\ker_{C(A)}(f)$ and $\operatorname{coker}_{C(A)}(f)$ are in $LCP(A)$.

Note that $X^m \xrightarrow{f} Y^m$ is a sum of morphisms $P(i)\langle m\rangle \to P(j)\langle m\rangle$,

and that $\operatorname{Hom}_{A\text{-grmod}}(P(i)\langle m\rangle, P(j)\langle m\rangle) = \operatorname{Hom}_{A\text{-grmod}}(Ae_i\langle m\rangle, Ae_j\langle m\rangle) = \begin{cases} 0 & i \neq j \\ k & i = j \end{cases}$

$\to$ so every morphism in here is either 0 or an iso. The claim now follows easily. $\square$

Note:

$$\begin{array}{ccccc} C(A\text{-grmod}) & \longrightarrow & K(A\text{-grmod}) & \longrightarrow & D(A\text{-grmod}) \\ \cup & & \cup & & \cup \\ LCP(A) & \xrightarrow{\cong} & LCP(A) & \xrightarrow{\cong} & LCP(A) \end{array}$$

Because we restricted to minimal complexes. — Standard fact about complexes of projectives

Answer to our main question:

$$\begin{array}{ccc} D^\uparrow(A\text{-grmod}) & \cong & D^\downarrow(A^!\text{-grmod}) \\ \cup & & \cup \\ \boxed{LCP(A)} & \xleftarrow{\cong} & A^!\text{-grmod} \end{array} \quad (\text{if } A \text{ is Koszul}).$$

Remarks: 1) $LCP(A) \cong A^!\text{-grmod}$ holds in general, we don't need A Koszul or even quadratic!

2)
$$X \longleftrightarrow M$$
$$X\langle m\rangle[m] \longleftrightarrow M\langle m\rangle$$

We'll now describe simple, projective and injective in $LCP(A)$

- Simple objects in $LCP(A)$ are $\mathcal{L}_i\langle m\rangle[-m]$: $\quad 0 \to \overset{-m}{P(i)\langle m\rangle} \to 0$
- Injective (indec.) objects, assuming A is Koszul $\quad (\cdots \to \overset{0}{P(i)} \to \overset{1}{0} \to \cdots)$

Consider the simple A-grmodule $L(i)$, and let Q_i be a linear projective resolution of it.

Claim: Q_i is an injective hull of $\mathcal{L}_i$ (Then Q_i are all indec inj objects up to shift ...)

Proof, assuming $\mathcal{L}_i$ has an injective hull I_i: $\forall X^\bullet \in LCP(A)$:

$\operatorname{Hom}_{LCP(A)}(X^\bullet, I_i) \cong \operatorname{Hom}_{A\text{-grmod}}(X^0, P(i)) \cong \operatorname{Hom}_k(X_0^0(i), k)$ naturally; So $Q_i \cong I_i$

and $\operatorname{Hom}_{LCP(A)}(X^\bullet, Q_i) \cong \operatorname{Hom}_{K(A\text{-grmod})}(X^\bullet, L(i))$

(Dually: if J_i is an linear injective resolution of $L(i)$, then $P_i = N^\bullet J_i$ is a projective cover of $\mathcal{L}_i$.)

- **Rem**: if A not Koszul, can introduce a functor $S: CP(A) \longrightarrow LCP(A)$ which "picks out the linear part".

not nec linear complexes of projectives

prop 17 in [MOS]

Then $I^\bullet_i := S Q^\bullet_i$ resp. $P^\bullet_i := S N^{-1} J^\bullet_i$ are inj hull resp proj cover of $\mathcal{L}_i$ in $LCP(A)$.
($Q^\bullet_i$: proj resol of $L(i)$; $J^\bullet_i$: inj resol of $L(i)$) (and these are all indec inj / proj).

In our examples: 1) $A = \mathbb{C}[x]$

$A^!$-gmod	$LCP(A)$
$L^!(1) = \mathbb{C}$	$\mathcal{L}^\bullet_1 = \cdots \longrightarrow 0 \longrightarrow \mathbb{C}[x] \longrightarrow 0 \longrightarrow \cdots$
$I^!(1) = \mathbb{C}[x]/(x^2)\langle -1\rangle$	$I^\bullet_1 = \cdots 0 \longrightarrow \mathbb{C}[x]\langle 1\rangle \xrightarrow{\cdot x} \mathbb{C}[x] \longrightarrow 0 \longrightarrow \cdots$ (the proj resol of $\mathbb{C}$)
$P^!(1) = \mathbb{C}[x]/(x^2)$	$P^\bullet_1 = \cdots \longrightarrow 0 \longrightarrow \mathbb{C}[x] \xrightarrow{\cdot x} \mathbb{C}[x]\langle -1\rangle \longrightarrow \cdots$

2) $A = A_Q$

$A^!$-gmod	$LCP(A)$
$L(1) = \langle e_1\rangle$	$\mathcal{L}^\bullet_1 = \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow P(1) \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$
$L(2) = \langle e_2\rangle$	$\mathcal{L}^\bullet_2 = \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow P(2) \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$
$I^!(2) = \langle a^*, e_2\rangle$	$I^\bullet_1 = \cdots \longrightarrow 0 \longrightarrow P(1)\langle 1\rangle \longrightarrow P(2) \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$
$I^!(1) = \langle (ba)^*, b^*, e_1\rangle \cong P^!(1)\langle -2\rangle$	$I^\bullet_2 = \cdots \longrightarrow P(1)\langle 2\rangle \longrightarrow P(2)\langle 1\rangle \longrightarrow P(1) \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$
$P^!(2) = \langle e_2, b\rangle$	$P^\bullet_1 = \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow P(2) \longrightarrow P(1)\langle -1\rangle \longrightarrow 0 \longrightarrow \cdots$
$P(1) = \langle e_1, a, ba\rangle$	$P^\bullet_2 = \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow P(1) \longrightarrow P(2)\langle -1\rangle \longrightarrow P(1)\langle -2\rangle \longrightarrow \cdots$

Using $LCP(A) \cong A^!$-gmod, one can prove the following

Theorem (Quadratic duality): For A as in our setup:

There is an adjoint pair of functors: $D^\downarrow(A\text{-gmod}) \underset{K'}{\overset{K}{\rightleftarrows}} D^\uparrow(A^!\text{-gmod})$

Satisfying:
- $K'(L^!(i)) \cong P(i)$
- $K(L(i)) \cong I(i)$
- $K'(X^\bullet\langle m\rangle[d]) \cong K(X^\bullet)(\langle m\rangle[d-m])$
- $K(X^\bullet\langle m\rangle[d]) \cong K(X^\bullet)(\langle m\rangle[d-m])$

Moreover: A is Koszul $\Leftrightarrow$ K, K' define an equivalence of categories

Examples of where Koszul rings/algebras appear "in nature"

1) **Category $\mathcal{O}$** (See [BGS, introduction])

Let $\mathfrak{g}$ be a semisimple complex Lie algebra; $\mathfrak{b}$ a Borel subalgebra.

$\mathcal{O}(\mathfrak{g}) := \left\{ \mathfrak{g}\text{-modules } M \;\middle|\; \begin{array}{l} M \text{ has weight space decomposition} \\ \text{any } m \in M \text{ lies in a finite dimensional } \mathfrak{b}\text{-submodule} \end{array} \right\}$

↳ "Nice category of $\mathfrak{g}$-modules, more general than just f.d. ones (e.g. contains all Verma modules), but still easier to understand than general $\mathfrak{g}$-modules."

$\mathcal{O}_0 = \{ M \in \mathcal{O} \mid (Z^+)^n M = 0 \text{ for } n \gg 0 \}$.

↳ augmentation ideal of $Z = (Z(U(\mathfrak{g}))$ (univ. enveloping algebra), i.e. $Z^+ = \mathrm{Ann}_Z \mathbb{C} \subset Z$

Set P be a projective generator of $\mathcal{O}_0$.

Then $\mathrm{End}_{\mathcal{O}}(P) \cong \mathrm{Ext}^{\bullet}_{\mathcal{O}}(L,L)$ is Koszul

and $\mathcal{O}_0 \cong \mathrm{Mod}_{fg}-(\mathrm{End}_{\mathcal{O}}(P))$

Example: $\mathfrak{g} = \mathfrak{sl}_2$

Then $\mathcal{O}_0$ contains the simple modules $L(0)$ (f.d.) and $L(-2)$ (∞ dim).

the Verma modules $M(-2) \cong L(-2)$ and $M(0)$: $L(-2) \hookrightarrow M(0) \twoheadrightarrow L(0)$

and the projective modules $P(0) \cong M(0)$ and $P(-2) \cong M(-1) \otimes L(1)$

↳ $M(0) \hookrightarrow P(-2) \twoheadrightarrow M(-2)$

$P = P(0) \oplus P(-2)$ is a projective generator of $\mathcal{O}_0$,

and $\mathrm{End}_{\mathcal{O}}(P)$ is isomorphic to the path algebra A of the quiver

$$\begin{tikzcd} \bullet \arrow[r, bend left, "a"] & \bullet \arrow[l, bend left, "b"] \end{tikzcd}$$

with relation $ba = 0$.

So $\mathcal{O}_0 \cong \mathrm{Mod}-A$